Derivation of the Lorentz Force Law and the Magnetic Field Concept using an Invariant Formulation of the Lorentz Transformation

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Abstract

It is demonstrated how the right hand sides of the Lorentz Transformation equations may be written, in a Lorentz invariant manner, as 4–vector scalar products. The formalism is shown to provide a short derivation, in which the 4–vector electromagnetic potential plays a crucial role, of the Lorentz force law of classical electrodynamics, and the conventional definition of the magnetic field in terms spatial derivatives of the 4–vector potential. The time component of the relativistic generalisation of the Lorentz force law is discussed. An important physical distinction between the space-time and energy-momentum 4–vectors is also pointed out.

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1 Introduction

Numerous examples exist in the literature of the derivation of electrodynamical equations from simpler physical hypotheses. In Einstein's original paper on Special Relativity [1], the Lorentz force law was derived by performing a Lorentz transformation of the electromagnetic fields and the space-time coordinates from the rest frame of an electron (where only electrostatic forces act) to the laboratory system where the electron is in motion and so also subjected to magnetic forces. A similar demonstration was given by Schwartz [2] who also showed how the electrodynamical Maxwell equations can be derived from the Gauss laws of electrostatics and magnetostatics by exploiting the 4-vector character of the electromagnetic current and the symmetry properties of the electromagnetic field tensor. The same type of derivation of electrodynamic Maxwell equations from the electrostatic and magnetostatic ones has recently been performed by the present author on the basis of 'space-time exchange symmetry' [3]. Frisch and Wilets [4] discussed the derivation of Maxwell's equations and the Lorentz force law by application of relativistic transforms to the electrostatic Gauss law. Dyson [5] published a proof, due originally to Feynman, of the Faraday-Lenz law of induction, based on Newton's Second Law and the quantum commutation relations of position and momentum, that excited considerable interest and a flurry of comments and publications [6, 7, 8, 9, 10, 11] about a decade ago. Landau and Lifshitz [12] presented a derivation of Ampère's Law from the electrodynamic Lagrangian, using the Principle of Least Action. By relativistic transformation of the Coulomb force from the rest frame of a charge to another inertial system in relative motion, Lorrain, Corson and Lorrain [13] derived both the Biot-Savart law, for the magnetic field generated by a moving charge, and the Lorentz force law.

In many text books on classical electrodynamics the question of what are the fundamental physical hypotheses underlying the subject, as distinct from purely mathematical developments of these hypotheses, used to derive predictions, is not discussed in any detail. Indeed, it may even be stated that it is futile to address the question at all. For example, Jackson [14] states:

At present it is popular in undergraduate texts and elsewhere to attempt to derive magnetic fields and even Maxwell equations from Coulomb's law of electrostatics and the theory of Special Relativity. It should immediately obvious that, without additional assumptions, this is impossible.'

This is, perhaps, a true statement. However, if the additional assumptions are weak ones, the derivation may still be a worthwhile exercise. In fact, in the case of Maxwell's equations, as shown in References [2, 3], the 'additional assumptions' are merely the formal definitions of the electric and magnetic fields in terms of the space–time derivatives of the 4–vector potential [15]. In the case of the derivation of the Lorentz force equation given below, not even the latter assumption is required, as the magnetic field definition appears naturally in the course of the derivation.

In the chapter on 'The Electromagnetic Field' in Misner Thorne and Wheeler's book 'Gravitation' [16] can be found the following statement:

Here and elsewhere in science, as stressed not least by Henri Poincaré, that view is

out of date which used to say, "Define your terms before you proceed". All the laws and theories of physics, including the Lorentz force law, have this deep and subtle chracter, that they both define the concepts they use (here $\vec{B}$ and $\vec{E}$) and make statements about these concepts. Contrariwise, the absence of some body of theory, law and principle deprives one of the means properly to define or even use concepts. Any forward step in human knowlege is truly creative in this sense: that theory concept, law, and measurement —forever inseperable—are born into the world in union.

I do not agree that the electric and magnetic fields are the fundamental concepts of electromagnetism, or that the Lorentz force law cannot be derived from simpler and more fundamental concepts, but must be 'swallowed whole', as this passage suggests. As demonstrated in References [2, 3] where the electrodynamic and magnetodynamic Maxwell equations are derived from those of electrostatics and magnetostatics, a more economical description of classical electromagentism is provided by the 4–vector potential. Another example of this is provided by the derivation of the Lorentz force law presented in the present paper. The discussion of electrodynamics in Reference [16] is couched entirely in terms of the electromagnetic field tensor, $F^{\mu\nu}$, and the electric and magnetic fields which, like the Lorentz force law and Maxwell's equations, are 'parachuted' into the exposition without any proof or any discussion of their interrelatedness. The 4–vector potential is introduced only in the next-but-last exercise at the end of the chapter. After the derivation of the Lorentz force law in Section 3 below, a comparison will be made with the treatment of the law in References [2, 14, 16].

The present paper introduces, in the following Section, the idea of an 'invariant formulation' of the Lorentz Transformation (LT) [17]. It will be shown that the RHS of the LT equations of space and time can be written as 4-vector scalar products, so that the transformed 4-vector components are themselves Lorentz invariant quantities. Consideration of particular length and time interval measurements demonstrates that this is a physically meaningful concept. It is pointed out that, whereas space and time intervals are, in general, physically independent physical quantities, this is not the case for the space and time components of the energy-momentum 4-vector. In Section 3, a derivation of the Lorentz force law, and the associated magnetic field concept, is given, based on the invariant formulation of the LT. The derivation is very short, the only initial hypothesis being the usual definition of the electric field in terms of the 4-vector potential, which, in fact, is also uniquely specified by requiring the definition to be a covariant one. In Section 4 the time component of Newton's Second Law in electrodynamics, obtained by applying space-time exchange symmetry [3] to the Lorentz force law, is discussed.

Throughout this paper it is assumed that the electromagnetic field constitutes, together with the moving charge, a conservative system; i.e. effects of radiation, due to the acceleration of the charge, are neglected

2 Invariant Formulation of the Lorentz Transformation

The space-time LT equations between two inertial frames S and S', written in a space-

time symmetric manner, are:

$$x' = \gamma(x - \beta x^0) \tag{2.1}$$
$$y' = y \tag{2.2}$$
$$z' = z \tag{2.3}$$
$$x'^0 = \gamma(x^0 - \beta x) \tag{2.4}$$

The frame S' moves with velocity, v, relative to S, along the common x-axis of S and S'. β and γ are the usual relativistic parameters:

$$\beta \equiv \frac{v}{c} \tag{2.5}$$
$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \tag{2.6}$$

where c is the speed of light, and

$$x^0 \equiv ct \tag{2.7}$$

where t is the time recorded by an observer at rest in S. Clocks in S and S' are synchronised, i.e., $t = t' = 0$, when the origins of the spatial cordinates of S and S' coincide.

Eqns(2.1)-(2.4) give the relation between space and time intervals Δx, $\Delta x^0 = c\Delta t$ as observed in the two frames:

$$\Delta x' = \gamma(\Delta x - \beta \Delta x^0) \tag{2.8}$$
$$\Delta y' = \Delta y \tag{2.9}$$
$$\Delta z' = \Delta z \tag{2.10}$$
$$\Delta x'^0 = \gamma(\Delta x^0 - \beta \Delta x) \tag{2.11}$$

Each interval can be interpreted as the result of a particular measurement performed in the appropriate frame. For example, Δx may correspond to the measurement of the distance between two points lying along the x-axis at a fixed time in S. In virtue of this, it may be identified with the space-like invariant interval, S_x, where:

$$S_x \equiv \sqrt{-(\Delta x^0)^2 + \Delta x^2} = \Delta x \tag{2.12}$$

since, for the measurement procedure just described, $\Delta x^0 = 0$. Notice that Δx is not *necessarily* defined in terms of such a measurement. If, following Einstein [1], the interval Δx is associated with the length, ℓ, of a measuring rod at rest in S and lying parallel to the x-axis, measurements of the ends of the rod can be made at arbitarily different times in S. The same result $\ell = \Delta x$ will be found for the length of the rod, but the corresponding invariant interval, S_x, as defined by Eqn(2.12) will be different in each case. Similarly, Δx^0 may be identified with the time-like invariant interval corresponding to successive observations of a clock at a fixed position (i.e. $\Delta x = 0$) in S:

$$S_0 \equiv \sqrt{(\Delta x^0)^2 - \Delta x^2} = \Delta x^0 \tag{2.13}$$

The interval Δx^0 could also be measured by observing the difference of the times recorded by a local clock and another, synchronised, one located at a different position in S, after a suitable correction for light propagation time delay. Each such pair of clocks would yield

the same value, Δx^0, for the time difference between two events in S, but with different values of the invariant interval defined by Eqn(2.13).

In virtue of Eqns(2.12) and (2.13) the LT equations (2.8) and (2.11) may be written the following *invariant* form:

$$S'_x = -\bar{U}(\beta) \cdot S \tag{2.14}$$
$$S'_0 = U(\beta) \cdot S \tag{2.15}$$

where the following 4–vectors have been introduced:

$$S \equiv (S_0; S_x, 0, 0) = (\Delta x^0; \Delta x, 0, 0) \tag{2.16}$$
$$U(\beta) \equiv (\gamma; \gamma\beta, 0, 0) \tag{2.17}$$
$$\bar{U}(\beta) \equiv (\gamma\beta; \gamma, 0, 0) \tag{2.18}$$

The time-like 4-vector, U, is equal to V/c, where V is the usual 4–vector velocity, whereas the space-like 4–vector, $\bar{U}$, is 'orthogonal to U in four dimensions':

$$U(\beta) \cdot \bar{U}(\beta) = 0 \tag{2.19}$$

Since the RHS of (2.14) and (2.15) are 4–vector scalar products, S'_x and S'_0 are manifestly Lorentz invariant quantites. These 4–vector components may be defined, in terms of specific space-time measurements, by equations similar to (2.12) and (2.13) in the frame S'. Note that the 4–vectors S and S' are 'doubly covariant' in the sense that $S \cdot S$ and $S' \cdot S'$ are 'doubly invariant' quantities whose spatial and temporal terms are, individually, Lorentz invariant:

$$S \cdot S = S_0^2 - S_x^2 = S' \cdot S' = (S'_0)^2 - (S'_x)^2 \tag{2.20}$$

Every term in Eqn(2.20) remains invariant if the spatial and temporal intervals described above are observed from a third inertial frame S" moving along the x-axis relative to both S and S'. This follows from the manifest Lorentz invariance of the RHS of Eqn(2.14) and (2.15) and their inverses:

$$S_x = -\bar{U}(-\beta) \cdot S' \tag{2.21}$$
$$S_0 = U(-\beta) \cdot S' \tag{2.22}$$

Since the LT Eqns(2.1) and (2.4) are valid for any 4–vector, W, it follows that:

$$W'_x = -\bar{U}(\beta) \cdot W \tag{2.23}$$
$$W'_0 = U(\beta) \cdot W \tag{2.24}$$

Again, W'_x and W'_0 are manifestly Lorentz invariant. An interesting special case is the energy-momentum 4–vector, P, of a physical object of mass, m. Here the 'doubly invariant' quantity analagous to $S \cdot S$ in Eqn(2.20) is equal to m^2c^2. Choosing the x-axis parallel to $\vec{p}$ and β to correspond to the object's velocity, so that S' is the object's proper frame, and since $P \equiv mcU(\beta)$, Eqns(2.23) and (2.24) yield, for this special case:

$$P'_x = -mc\bar{U}(\beta) \cdot U(\beta) = 0 \tag{2.25}$$
$$P'_0 = mcU(\beta) \cdot U(\beta) = mc \tag{2.26}$$

Since the Lorentz transformation is determined by the single parameter, β, then it follows from Eqns(2.25) and (2.26) that, unlike in the case of the space and time intervals in Eqns(2.8) and (2.11), the spatial and temporal components of the energy momentum 4–vector, in an arbitary inertial frame, are not independent. In fact, P_0 is determined in terms of P_x and m by the relation, that follows from the inverse of Eqns(2.25) and (2.26): $P_0 = \sqrt{P_x^2 + m^2c^2}$. Thus, although the LT equations for the space-time and energy-momentum 4–vectors are mathematically identical, the physical interpretation of the transformed quantities is quite different in the two cases.

The LT equation, (2.24), for the electromagnetic 4–vector potential, A, is found to play a crucial role in the derivation of the Lorentz force law presented in the following Section.

3 Derivation of the Lorentz force law and the Magnetic Field

In electrostatics, the electric field, $\vec{E}$, is customarily written in terms of the electrostatic potential, ϕ, according to the equation $\vec{E} = -\vec{\nabla}\phi$. The potential at a distance, r, from a point charge, Q, is given by Coulomb's law $\phi(r) = Q/r$. This, together with the equation $\vec{F} = q\vec{E}$, defining the force, $\vec{F}$, exerted on a charge, q, by the electric field, completes the specification of the dynamical basis of classical electromagnetism.

It remains to generalise the above equation relating the electric field to the electrostatic potential in a manner consistent with special relativity. In relativistic notation [18], the electric field is related to the potential by the equation: $E^i = \partial^i A^0$, where ϕ is identified with the time component, A^0, of the 4–vector electromagnetic potential $(A^0;\vec{A})$. In order to respect special relativity the electric field must be defined in a covariant manner, i.e. in the same way in all inertial frames. The electrostatic law may be generalised in two ways:

$$E^i \rightarrow E^i_{\pm} \equiv \partial^i A^0 \pm \partial^0 A^i \tag{3.1}$$

This equation shows the only possiblities to define the electric field in a way that respects the symmetry with respect to the exchange of space and time coordinates that is a general property of all special relativistic laws [3]. Choosing $i = 1$ in Eqn(3.1) and transforming all quantities on the RHS into the S' frame, by use of the inverses of Eqns(2.1) and (2.4), leads to the following expressions for the 1–component of the electric field in S, in terms of quantites defined in S':

$$E^1_{\pm} = \gamma^2(1 \pm \beta^2)\partial'^1 A'^0 + \gamma^2(\beta^2 \pm 1)\partial'^0 A'^1 + \gamma^2\beta(1 \pm 1)(\partial'^0 A'^0 + \partial'^1 A'^1) \tag{3.2}$$

Only the choice $E^1 \equiv E^1_-$ yields a covariant definition of the electric field. In this case, using Eqns(2.5) and (2.6), Eqn(3.2) simplifies to:

$$E^1 = \partial'^1 A'^0 - \partial'^0 A'^1 = E'^1 \tag{3.3}$$

Which expresses the well-known invariance of the longitudinal component of the electric field under the LT.

Thus, from rotational invariance, the general covariant definition of the electric field is:

$$E^i = \partial^i A^0 - \partial^0 A^i \tag{3.4}$$

This is the 'additional assumption', mentioned by Jackson in the passage quoted above, that is necessary, in the present case, to derive the Lorentz force law. However, as written, it concerns only the physical properties of the electric field: the magnetic field concept has not yet been introduced. A further *a posteriori* justification of Eqn(3.4) will be given after derivation of the Lorentz force law. Here it is simply noted that, if the spatial part of the 4–vector potential is time-independent, Eqn(3.4) reduces to the usual electrostatic definition of the electric field.

The force $\vec{F}'$ on an electric charge q at rest in the frame S' is given by the definition of the electric field, and Eqn(3.4) as:

$$F'^i = q(\partial'^i A'^0 - \partial'^0 A'^i) \tag{3.5}$$

Equations analagous to (2.24) may be written relating A' and ∂' to the corresponding quantities in the frame S moving along the x' axis with velocity $-v$ relative to S':

$$\partial'^0 = U(\beta) \cdot \partial \tag{3.6}$$
$$A'^0 = U(\beta) \cdot A \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.5) gives:

$$F'^i = q\left[\partial'^i (U(\beta) \cdot A) - (U(\beta) \cdot \partial) A'^i\right] \tag{3.8}$$

This equation expresses a linear relationship between F'^i, ∂'^i and A'^i. Since the coefficients of the relation are Lorentz invariant, the same formula is valid in any inertial frame, in particular, in the frame S. Hence:

$$F^i = q\left[\partial^i (U(\beta) \cdot A) - (U(\beta) \cdot \partial) A^i\right] \tag{3.9}$$

This equation gives, in 4–vector notation, a spatial component of the Lorentz force on the charge q in the frame S, and so completes the derivation.

To express the Lorentz force formula in the more familiar 3-vector notation, it is convenient to introduce the relativistic generalisation of Newton's Second Law [19]:

$$\frac{dP}{d\tau} = F \tag{3.10}$$

where F is the 4-vector force and $\tau = t'$ is the proper time (in S') that is related to the time t in S by the relativistic time dilatation formula: $dt = \gamma d\tau$. This gives, with Eqn(3.9) and (3.10):

$$\begin{aligned} \frac{dP^i}{d\tau} &= \gamma \frac{dP^i}{dt} \\ &= q(\partial^i A^\alpha - \partial^\alpha A^i) U(\beta)_\alpha \\ &= \gamma q\left[\partial^i A^0 - \partial^0 A^i - \beta_j(\partial^i A^j - \partial^j A^i) - \beta_k(\partial^i A^k - \partial^k A^i)\right] \end{aligned} \tag{3.11}$$

Introducing now the *magnetic field* according to the definition [20]:

$$B^k \equiv -\epsilon_{ijk}(\partial^i A^j - \partial^j A^i) = (\vec{\nabla} \times \vec{A})^k \tag{3.12}$$

enables Eqn(3.11) to be written in the compact form:

$$\frac{dP^i}{dt} = q\left[E^i + \beta_j B^k - \beta_k B^j\right] = q\left[E^i + (\vec{\beta} \times \vec{B})^i\right] \tag{3.13}$$

so that, in 3–vector notation, the Lorentz force law is:

$$\frac{d\vec{p}}{dt} = mc\frac{d\gamma\vec{\beta}}{dt} = q[\vec{E} + \vec{\beta} \times \vec{B}] \tag{3.14}$$

Writing Eqn(3.4) in 3–vector notation and performing vector multiplication of both sides by the differential operator $\vec{\nabla}$ gives:

$$\vec{\nabla} \times \vec{E} = (\vec{\nabla} \times \vec{\nabla})A^0 - \partial^0(\vec{\nabla} \times \vec{A}) = -\frac{\partial \vec{B}}{\partial t} \tag{3.15}$$

where Eqn(3.12) has been used. Eqn(3.15) is just the Faraday-Lenz induction law, i.e. the magnetodynamic Maxwell equation. This is only apparent, however, once the 'magnetic field' concept of Eqn(3.12) has been introduced. Thus the initial hypothesis, Eqn(3.4), is actually a Maxwell equation. This is the *a posteriori* justification, mentioned above, for this covariant definition of the electric field.

It is common in discussions of electromagnetism to introduce the second rank electromagnetic field tensor, $F^{\mu\nu}$ according to the definition:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.16}$$

in terms of which, the electric and magnetic fields are defined as:

$$E^i \equiv F^{i0} \tag{3.17}$$

$$B^k \equiv -\epsilon_{ijk}F^{ij} \tag{3.18}$$

From the point of view adopted in the present paper both the electromagnetic field tensor and the electric and magnetic fields themselves are auxiliary quantities introduced only for mathematical convenience, in order to write the equations of electromagnetism in a compact way. Since all these quantities are completly defined by the 4–vector potential, it is the latter quantity that encodes all the relevant physical information on any electrodynamic problem [21]. This position is contrary to that commonly taken in the literature and texbooks where it is often claimed that only the electric and magnetic fields have physical significance, while the 4–vector potential is only a convenient mathematical tool. For example Röhrlich [22] makes the statement:

These functions (ϕ and $\vec{A}$) known as potentials have no physical meaning and are introduced solely for the purpose of mathematical simplification of the equations.

In fact, as shown above (compare Eqns(3.11) and (3.13)) it is the introduction of the electric and magnetic fields that enable the Lorentz force equation to be written in a simple

manner! In other cases (e.g. Maxwell's equations) simpler expessions may be written in terms of the 4–vector potential. The quantum theory, quantum electrodynamics, that underlies classical electromagnetism, requires the introduction the 4–vector photon field $\mathcal{A}^\mu$ in order to specify the minimal interaction that provides the dynamical basis of the theory. Similarly, the introduction of A^μ is necessary for the Lagrangian formulation of classical electromagnetism. It makes no sense, therefore, to argue that a physical concept of such fundamental importance has 'no physical meaning'.

The initial postulate used here to derive the Lorentz force law is Eqn(3.4), which contains, explicitly, the electrostatic force law and, implicitly, the Faraday-Lenz induction law. The actual form of the electrostatic force law (Coulomb's inverse square law) is not invoked, suggesting that the Lorentz force law may be of greater generality. On the assumption of Eqn(3.4) (which has been demonstrated to be the only possible covariant definition of the electric field), the existence of the 'magnetic field', the 'electromagnetic field tensor', and finally the Lorentz force law itself have all been derived, without further assumptions, by use of the invariant formulation of the Lorentz transformation.

It is instructive to compare the derivation of the Lorentz force law given in the present paper with that of Reference [13] based on the relativistic transformation properties of the Coulomb force 3–vector. Coulomb's law is not used in the present paper. On the other hand, Reference [13] makes no use of the 4–vector potential concept, which is essential for the derivation presented here. This demonstrates an interesting redundancy among the fundamental physical concepts of classical electromagnetism.

In Reference [2], Eqns(3.4), (3.12) and (3.16) were all introduced as *a priori* initial postulates without further justification. In fact, Schwartz gave the following explanation for his introduction of Eqn(3.16) [23]:

So far everything we have done has been entirely deductive, making use only of Coulomb's law, conservation of charge under Lorentz transformation and Lorentz invariance for our physical laws. We have now come to the end of this deductive path. At this point when the laws were being written, God had to make a decision. In general there are 16 components of a second-rank tensor in four dimensions. However, in analogy to three dimensions we can make a major simplification by choosing the completely antisymmetric tensor to represent our field quantities. Then we would have only 6 independent components instead of the possible 16. Under Lorentz transformation the tensor would remain antisymmetric and we would never have need for more than six independent components. Appreciating this, and having a deep aversion to useless complication, God naturally chose the antsymmetric tensor as His medium of expression.

Actually it is possible that God may have previously invented the 4–vector potential and special relativity, which lead, as shown above, to Eqn(3.4) as the only possible covariant definition of the electric field. As also shown in the present paper, the existence of the remaining elements of the antisymmetric field tensor, containing the magnetic field, then follow from special relativity alone. Schwartz derived the Lorentz force law, as in Einstein's original Special Relativity paper [1], by Lorentz transformation of the electric field, from the rest frame of the test charge, to one in which it is in motion. This requires that the magnetic field concept has previously been introduced as well as knowledge of the Lorentz transformation laws of the electric and magnetic fields.

In the chapter devoted to special relativity in Jackson's book [24] the Lorentz force law is simply stated, without any derivation, as are also the defining equations of the electric and magnetic fields and the electromagnetic field tensor just mentioned. No emphasis is therefore placed on the fundamental importance of the 4–vector potential in the relativistic description of electromagnetism.

In order to treat, in a similar manner, the electromagnetic and gravitational fields, the discussion in Misner Thorne and Wheeler [16] is largely centered on the properties of the tensor $F^{\mu\nu}$. Again the Lorentz force equation is introduced, in the spirit of the passage quoted above, without any derivation or discussion of its meaning. The defining equations of the electric and magnetic fields and $F^{\mu\nu}$, in terms of A^{μ}, appear only in the eighteenth exercise of the relevant chapter. The main contents of the chapter on the electromagnetic field are an extended discussion of purely mathematical tensor manipulations that obscure the essential simplicity of electromagnetism when formulated in terms of the 4–vector potential.

In contrast to References [2, 24, 16], in the derivation of the Lorentz force law and the magnetic field presented here, the only initial assumption, apart from the validity of special relativity, is the chosen definition, Eqn(3.4), of the electric field in terms of the 4–vector potential A^{μ}, which is the only covariant one. Thus, a more fundamental description of electromagnetism than that provided by the electric and magnetic field concepts is indeed possible, contrary to the opinion expressed in the passage from Misner Thorne and Wheeler quoted above.

4 The time component of Newton's Second Law in Electrodynamics

Space-time exchange symmetry [3] states that physical laws in flat space are invariant with respect to the exchange of the space and time components of 4-vectors. For example, the LT of time, Eqn(2.4), is obtained from that for space, Eqn(2.1), by applying the space-time exchange (STE) operations: $x_0 \leftrightarrow x$, $x'_0 \leftrightarrow x'$. In the present case, application of the STE operation to the spatial component of the Lorentz force equation in the second line of Eqn(3.11) leads to the relation:

$$\begin{aligned} \frac{dP^0}{d\tau} &= \frac{\gamma}{c}\frac{dP^0}{dt} = q(\partial^0 A^{\alpha} - \partial^{\alpha} A^0)U(\beta)_{\alpha} \\ &= -qE^i U(\beta)_i = \gamma q \frac{\vec{E}\cdot\vec{v}}{c} \end{aligned} \tag{4.1}$$

where Eqns(2.5) and (3.4) and the following properties of the STE operation [3] have been used:

$$\partial^0 \leftrightarrow -\partial^i \tag{4.2}$$

$$A^0 \leftrightarrow -A^i \tag{4.3}$$

$$C\cdot D \leftrightarrow -C\cdot D \tag{4.4}$$

Eqn(4.1) yields an expression for the time derivative of the relativistic energy, $\mathcal{E} = P^0$:

$$\frac{d\mathcal{E}}{dt} = q\vec{E}\cdot\vec{v} = q\vec{E}\cdot\frac{d\vec{x}}{dt} \tag{4.5}$$

Integration of Eqn(4.5) gives the equation of energy conservation for a particle moving from an initial position, $\vec{x}_I$, to a final position, $\vec{x}_F$, under the influence of electromagnetic forces:

$$\int_{\mathcal{E}_I}^{\mathcal{E}_F} d\mathcal{E} = q\int_{\vec{x_I}}^{\vec{xF}} \vec{E}\cdot d\vec{x} \tag{4.6}$$

Thus work is done on the moving charge only by the electric field. This is also evident from the Lorentz force equation, (3.14), since the magnetic force $\simeq \vec{\beta}\times\vec{B}$ is perpendicular to the velocity vector, so that no work is performed by the magnetic field. A corollary is that the relativistic energy (and hence the magnitude of the velocity) of a charged particle moving in a constant magnetic field is a constant of the motion. Of course, Eqn(4.5) may also be derived directly from the Lorentz force law, so that the time component of the relativistic generalisation of Newton's Second Law, Eqn(4.1), contains no physical information not already contained in the spatial components. This is related to the fact that, as demonstrated in Eqns(2.25) and (2.26), the spatial and temporal components of the energy-momentum 4–vector are not independent physical quantities.

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[14] J.D.Jackson, 'Classical Electrodynamics', (John Wiley and Sons, New York, 1975) Section 12.2, P578.

[15] Actually, a careful examination of the derivation of Ampère's from the Gauss law of electrostatics in Reference[3] shows that, although Eqn(3.4) of the present paper is a necessary initial assumption, the definition of the magnetic field in terms of the spatial derivatives of the 4–vector potential occurs naturally in the course of the derivation (see Eqns(5.16) and (5.17) of Reference[3]) so it is not necessary to assume, at the outset, the expression for the spatial components of the electromagnetic field tensor as given by Eqn(5.1) of Reference[3].

[16] C.W.Misner, K.S.Thorne and J.A.Wheeler, 'Gravitation', (W.H.Freeman, San Francisco, 1973) Ch 3, P71.

[17] This should not be confused with a manifestly *covariant* expression for the LT, where it is written as a linear 4-vector relation with Lorentz-invariant coefficients, as in: D.E.Fahnline, Am. J. Phys. **50** 818 (1982).

[18] A time-like metric is used for 4-vector products with the components of a 4–vector, W, defined as:

$$W_t = W^0 = W_0, \quad W_{x,y,z} = W^{1,2,3} = -W_{1,2,3}$$

and an implied summation over repeated contravariant (upper) and covariant (lower) indices. Repeated Greek indices are summed from 0 to 3, repeated Roman ones from 1 to 3. Also

$$\partial^\mu \equiv (\frac{\partial}{\partial x^0}; -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}) = (\partial^0; -\vec{\nabla})$$

[19] H.Goldstein, 'Classical Mechanics', (Addison-Wesley, Reading Massachusetts, 1959) P200, Eqn(6-30).

[20] The alternating tensor, ϵ_{ijk}, equals 1 (-1) for even (odd) permutations of ijk.

[21] The explicit form of A^μ, as derived from Coulomb's law, is given in standard text-books on classical electrodynamics. For example, in Reference[13], it is to be found in Eqns(17-51) and (17-52). A^μ is actually proportional to the 4-vector velocity, V, of the charged particle that is the source of the electromagnetic field.

[22] F.Röhrlich, 'Classical Charged Particles', (Addison-Wesley, Reading, MA, 1990) P65.

[23] Reference [2] above, Ch 3, P127.

[24] Reference [14] above, Section 11.9, P547.